Determining the exponent b using Linear Regression

To determine the exponent b, we use data from the disk sources at 7 energies: 3 energies from Bi^{207}, 2 energies from Na^{22}, and 2 energies from Co^{60}. The four parameters b, C_{Bi}, C_{Na} and C_{Co} need to be found that best fit the data. In this app we offer two ways to fit the data. One option is to use a formula for b, which we derive below using an unweighted linear regression approach for the logarithm of the data.

For the linear regression approach, one starts with the equations:

$$\begin{aligned}
(C/Y)_i &= C_{Bi}(E_i/1460)^b \qquad (i = 1, 2, 3)\\
(C/Y)_i &= C_{Na}(E_i/1460)^b \qquad (i = 4, 5)\\
(C/Y)_i &= C_{Co}(E_i/1460)^b \qquad (i = 6, 7)
\end{aligned}$$

Note that the exponent b is the same for all the sources. We need to vary C_{Bi}, C_{Na}, C_{Co} and b for a "best fit" to the data.

To linearize the equations, one takes the log of both sides of the equations:

$$\begin{aligned}
ln((C/Y)_i) &= b\, ln(E_i/1460) + ln(C_{Bi}) \qquad (i = 1, 2, 3)\\
ln((C/Y)_i) &= b\, ln(E_i/1460) + ln(C_{Na}) \qquad (i = 4, 5)\\
ln((C/Y)_i) &= b\, ln(E_i/1460) + ln(C_{Co}) \qquad (i = 6, 7)
\end{aligned}$$

We define $y_i \equiv ln((C/Y)_i)$ for the count data, $x_i \equiv ln(E_i/1460)$ for the energy data, $k_{Bi} \equiv ln(C_{Bi})$, $k_{Na} \equiv ln(C_{Na})$, and $k_{Co} \equiv ln(C_{Co})$. Then the equations become:

$$\begin{aligned}
y_i &= b\, x_i + k_{Bi} \qquad (i = 1, 2, 3)\\
y_i &= b\, x_i + k_{Na} \qquad (i = 4, 5)\\
y_i &= b\, x_i + k_{Co} \qquad (i = 6, 7)
\end{aligned}$$

There are 4 free parameters to vary to best fit the data: b, k_{Bi}, k_{Na}, and k_{Co}. To determine the "best fit" values we use linear regression. The chi-square function χ^2 is defined as:

$$\chi^2 \equiv \sum_{i=1}^{3}(b\, x_i + k_{Bi} - y_i)^2 + \sum_{i=4}^{5}(b\, x_i + k_{Na} - y_i)^2 + \sum_{i=6}^{7}(b\, x_i + k_{Co} - y_i)^2 \quad (1)$$

The "best fit" values are the ones that minimize the χ^2 function. At the minimum value of χ^2, the derivatives with respect to each of the free parameters are zero: $\frac{\partial \chi^2}{\partial b} = 0$, $\frac{\partial \chi^2}{\partial k_{Bi}} = 0$, $\frac{\partial \chi^2}{\partial k_{Na}} = 0$, and $\frac{\partial \chi^2}{\partial k_{Co}} = 0$.

The derivative with respect to b yields:

$$\frac{\partial \chi^2}{\partial b} = \sum_{i=1}^{3} 2(b\,x_i + k_{Bi} - y_i)x_i + \sum_{i=4}^{5} 2(b\,x_i + k_{Na} - y_i)x_i + \sum_{i=6}^{7} 2(b\,x_i + k_{Co} - y_i)x_i = 0 \tag{2}$$

The above equation can be written as

$$bX^2 + k_{Bi}X_{Bi} + k_{Na}X_{Na} + k_{Co}X_{Co} - \sum_{i=1}^{7}(y_i x_i) = 0 \tag{3}$$

where $X^2 \equiv \sum_{i=1}^{7} x_i^2$, $X_{Bi} \equiv \sum_{i=1}^{3} x_i$, $X_{Na} \equiv \sum_{i=4}^{5} x_i$, $X_{Co} \equiv \sum_{i=6}^{7} x_i$. The derivative with respect to k_{Bi} yields:

$$\frac{\partial \chi^2}{\partial k_{Bi}} = \sum_{i=1}^{3} 2(b\,x_i + k_{Bi} - y_i) = 0 \tag{4}$$

$$bX_{Bi} + 3k_{Bi} - Y_{Bi} = 0 \tag{5}$$

where $Y_{Bi} \equiv \sum_{i=1}^{3} y_i$. Similarily, for the last two parameters we have

$$bX_{Na} + 2k_{Na} - Y_{Na} = 0 \tag{6}$$

$$bX_{Co} + 2k_{Co} - Y_{Co} = 0 \tag{7}$$

with $Y_{Na} \equiv \sum_{i=4}^{5} y_i$ and $Y_{Co} \equiv \sum_{i=6}^{7} y_i$
Combining equations 3, 5, 6, and 7, we obtain one equation involving only b:

$$b(X^2 - \frac{X_{Bi}^2}{3} - \frac{X_{Na}^2}{2} - \frac{X_{Co}^2}{2}) - \sum_{i=1}^{7}(x_i y_i) = -\frac{Y_{Bi}X_{Bi}}{3} - \frac{Y_{Na}X_{Na}}{2} - \frac{Y_{Co}X_{Co}}{2} \tag{8}$$

Which can be solved for b

$$b = \frac{X^2 - Y_{Bi}X_{Bi}/3 - Y_{Na}X_{Na}/2 - Y_{Co}X_{Co}/2}{\sum_{i=1}^{7}(x_i y_i) - (X_{Bi})^2/3 - (X_{Na})^2/2 - (X_{Co})^2/2} \tag{9}$$

Once b is determined, one can solve for the other three parameters:

$$\begin{aligned} k_{Bi} &= (Y_{Bi} - bX_{Bi})/3 \\ k_{Na} &= (Y_{Na} - bX_{Na})/2 \\ k_{Co} &= (Y_{Co} - bX_{Co})/2 \end{aligned}$$

and then the three C_D values. The formulas above are a minor generalization of the linear regression formula commonly used in undergraduate physics labs.