Transcendental Measurementation

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Transcendental numbers such as pi and the natural logarithm e are introduced in mathematics classes, where their mathematical origins and series expansions are presented. However, these numbers find their way into physics equations, where quite often they carry some physical significance. Usually they are taken to be their mathematical value and used to determine other physical quantities. Taking a different point of view, we thought it would be interesting to think of ways to directly measure transcendental as well as irrational numbers. That is, what physical quantities could we measure to determine π, e, and $\sqrt{2}$? For $\sqrt{2}$, the challenge was to determine its value without any measuring device, compass, or straightedge.

Measuring mathematically known numbers is not a trivial exercise. The students need to determine the essential physical conditions necessary to produce these special unitless numbers. Since measurements are made, uncertainties will result. Any differences in the expected mathematical values can lead to discussions about error analysis. If the measured values are systematically different from the expected ones, the students are challenged to figure out what went wrong. In this article, we present some ways in which we tried to measure π, e, and $\sqrt{2}$, and some of the difficulties we encountered.

Measuring $\sqrt{2}$

What happens to the motion of a system if all its dimensions are doubled? The answer to this question provides a method to measure $\sqrt{2}$ without using any measuring devices. The key physics is the following: for systems in which the potential energy is proportional to the size of the system, the time of the motion will scale as the square root of the spacial dimensions. This result comes from Newton's second law, which has a second derivative of position with respect to time. Projectile motion, objects rolling without slipping down hills, and pendulums, all without air friction, are examples of a few systems where this scaling applies. In the case of a simple pendulum of length l, the period of the motion is proportional to $\sqrt{l/g}$, a result that can also be obtained from dimensional analysis.[1] For a pendulum with a bob of radius r, the rotational motion of the bob needs to be included, and the period is proportional to $\sqrt{l\,[1 + \frac{2}{5}(r/l)^2]/g}$. Thus, examining the period of two pendulums, one exactly twice the size of the other, will allow a direct measurement of $\sqrt{2}$.

The procedure to measure $\sqrt{2}$ is as follows: a) set up two pendulums, one being twice the size of the other. Each pendulum will be observed by a different person. b) Both observers start their pendulums at the same time and count the number of oscillations over the same time interval. The $\sqrt{2}$ is the ratio of the number of counts of the two observers.

The pendulums can be set up as shown in Fig. 1, one with a spherical bob roughly twice the size of the other. First, the two are hung so the distances from the pivot point to the center of the bobs are the same. With this length marked on the one with the smaller bob, it is moved to another holder. Then the string is doubled over the pivot until it reaches the center of the bob (see Fig. 1b). This second pendulum then will have a length approximately one-half that of the first. To have a doubling of scale, the initial angle should be the same for both pendulums. We note that in the absence of air friction, the initial starting angle need not be small to obtain $\sqrt{2}$.

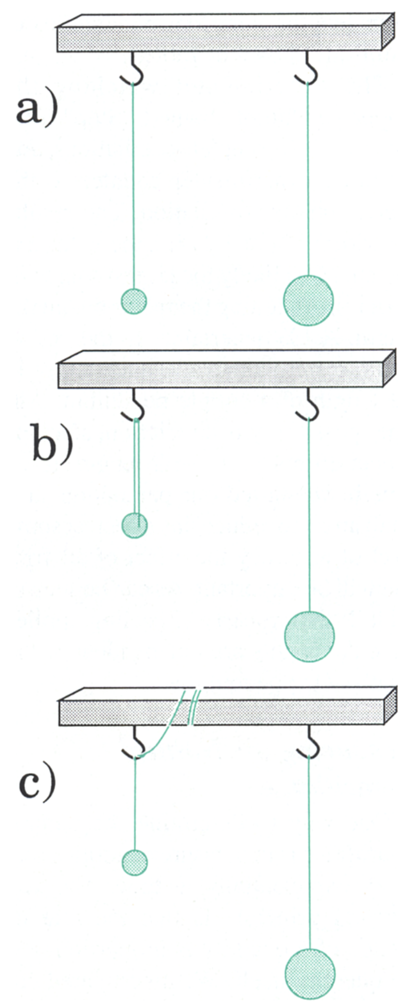

Fig. 1. Setup for constructing one pendulum to be half the size of another. First start them with the same length, then wrap the string of the smaller bob around the pivot till it comes to the center of the bob.

Table I.

Approximate length of long pendulum	Counts for long pendulum	Counts for short pendulum	Ratio
80 cm	100	141¾	1.4175
100 cm	100	143	1.43
120 cm	100	141	1.41
140 cm	100	142¼	1.4225
170 cm	100	141	1.41

We followed this procedure for different lengths of the long pendulum; our results are shown in Table I.

The experimenter watching the longer pendulum shouted "Stop" after counting 100 complete oscillations, and the other experimenter counted to the nearest quarter oscillation. The results are within 1% of 1.414..., the value expected. Most likely the largest source of error is in obtaining the factor of 2 in the length. A 1% uncertainty in the period would result from a 2% uncertainty in the length of a simple pendulum. Another possibility is air friction, since its effects do not scale with length. Even though we started our pendulums at a small angle to reduce this effect, at some level of accuracy the effect of air friction will be important. As can be seen in Table I, the property of scaling applies since the results are independent of the length of the pendulums.

Measuring the Natural Logarithm e

One way to determine the natural logarithm e is to measure quantities that decay exponentially in time. The key physical principle is that the rate of change of a quantity is proportional to the quantity itself. We investigated capacitor discharge and radioactive decay. In the case of capacitor discharge, we measured the current in an *RC* circuit at equal time intervals. A graph of the data can be used to illustrate how to determine e. In Fig. 2 we plot the current in a series *RC* plus ammeter circuit as a function of time. The value of C is 1 farad, R is 200 ohms, and readings of the current were taken every 10 seconds using a digital ammeter. The value of e can be found as follows: a) Pick any data point on the graph and construct a tangent to the curve; b) follow the tangent line down to the horizontal axis and mark the intersection point. The natural logarithm base e is (the current of the original point)/(the current of the intersection point). This can be seen by referring to Fig. 2. The decay of the current is described by

$$C(t) = C_i e^{-\lambda(t - t_i)}$$

where $C(t)$ is the current at time t, and C_i is the current at the time of the i'th reading, t_i. The slope of the tangent at t_i is given by

$$\left.\frac{dC(t)}{dt}\right|_{t = t_i} = -\lambda C_i$$

Thus we can deduce that the tangent line intercepts the t-axis at the value $t_i + 1/\lambda$. Using this result in the above equation yields

$$\frac{C_i}{C_x} = \frac{C_i}{C_i e^{-\lambda/\lambda}} = \frac{1}{e^{-1}} = \mathrm{e}$$

This procedure can be done numerically from the data, or by drawing the appropriate lines on the graph as shown in Fig. 2. This method is not perfect, since there is some systematic error associated with taking data at finite time intervals. The error is easily estimated to give a percent error in determining e of $\ln(\sqrt{2})\,\Delta T/\tau$, where ΔT is the time interval and τ is the half-life of the decay. For our capacitor decay experiments, this introduced an error of 1.7%.

Using the data from the capacitor discharge we determined e numerically from the measured currents. This can be done by first calculating the intersection point x, and then using linear interpolation to find the value of the current at this point. As before, we let C_i be the current of the i'th reading. Then, x can be obtained from the slope of C_i and C_{i+1}:

$$\frac{C_i}{x - i} = \frac{C_i - C_{i+1}}{1} \qquad (1)$$

$$x = i + \frac{C_i}{C_i - C_{i+1}}$$

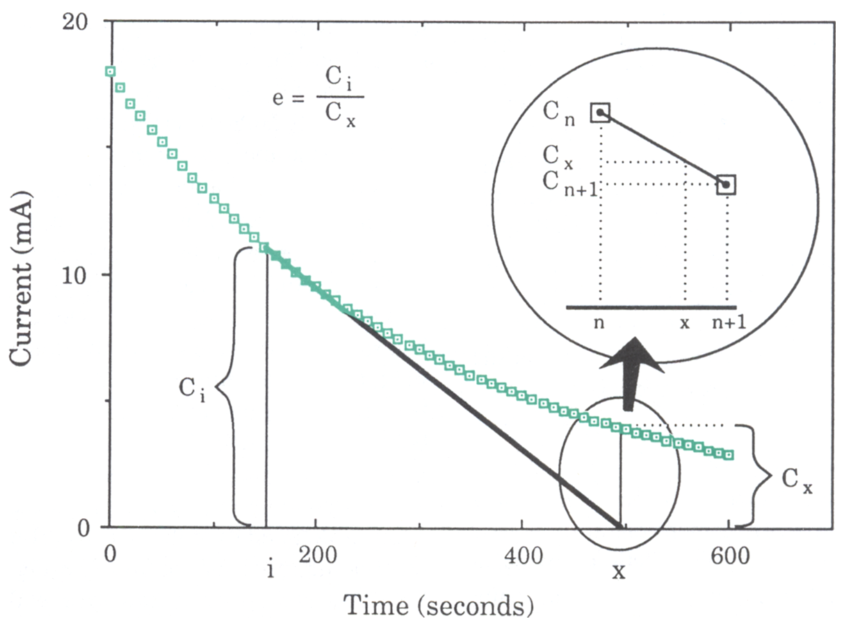

Fig. 2. Determining the natural logarithm e from the data.

The value x will usually lie between readings, and we can use linear interpolation to find an approximate value of the current at x. Let n be the n'th data point just before x, and C_x be the value of the current at x, as shown in Fig. 2. Then using similar triangles we have:

$$\frac{C_n - C_x}{C_n - C_{n+1}} = \frac{x-n}{1}$$

which gives

$$C_x = C_n + (x-n)(C_{n+1} - C_n) \quad (2)$$

Thus e is given by the expression

$$e = \frac{C_i}{C_x} = \frac{C_i}{C_n + (x-n)(C_{n+1} - C_n)} \quad (3)$$

A computer program was written to assist in the calculation of e from the data. The results of our measurements using different resistors are shown in Table II.

We calculated e for each i from $i = 1$ until $n + 1$ was larger than our last data point. The number of points used for each experiment is listed in column 2. The average value is shown in column 3 along with the standard deviation. We used two 150-Ω resistors, and investigated the affect of the ammeter on the first one. For this resistor, both ammeters gave similar results.

It is interesting to note that for the second 150-Ω resistor we obtained a value of e = 2.42 ± 0.06. What could have caused the result to be different from 2.71...? The final result does not depend on the value of R, the value of C, the initial charge on the capacitor, nor the internal resistance of the ammeter. If the ammeter is off by a scaling factor, this should also not effect our final answer for e. It is these insensitivities to the systems parameters that make the number e so special. The only thing that can cause a different value for e are nonlinearities in the circuit. If the resistance or capacitance depends on the current, then the differential equation describing the circuit will be nonlinear. For example, a possible nonlinear contribution would be a capacitor leaking with a current that is not proportional to the charge. Any difference in the measurement of e from 2.71... will thus be a measure of the amount of nonlinearity in the circuit, which in our case was less than 2%. Thus it is important that the ammeter be kept on the same scale throughout the entire experiment. We believe that the problem lies with the resistor we used, since for other resistors our values for e lie within the expected value.

Table II.

Experiment		Natural Log e and standard deviation
Capacitor Discharge C = 1 farad	**Data points used**	
R = 100 Ω	16	2.75 ± 0.04
R = 150 Ω:		
Fluke Ammeter	29	2.66 ± 0.08
Beckman Ammeter	28	2.73 ± 0.06
R = 200 Ω	27	2.65 ± 0.11
R = 150 Ω	17	2.42 ± 0.06
Radioactive Decay of Barium 137		**2.55 to 2.74**

For the radioactive decay, we examined the decay of ^{137}Ba from the common "Cesium Cow" experiment often used in the classroom. The data were not as smooth as in the capacitor discharge case due to statistical fluctuations, and we had to use the graphical method to obtain an approximate value for e. Our values ranged between 2.55 and 2.74.

Measuring π

Measuring π is very easy, since it is just the ratio of the circumference to the diameter of a circle. This well-known exercise is probably best suited for a seventh-, eighth-, or ninth-grade class. The students measure the circumference and diameter of an assortment of circular objects. The data can be analyzed in various ways. You could graph the diameter D on the x-axis and the circumference C on the y-axis and use this experiment as an introduction to graphing. For more advanced classes, errors in D and C can be included in the graph and uncertainties in determining the slope discussed. The students could also divide C by D for each object, and then calculate an average value and standard deviation for π. Another possibility is to use this experiment to present propagation of error. Letting δC represent the uncertainty in measuring C and δD the uncertainty in D, the uncertainty in π is given by $\delta\pi = \pi\,[\delta D/D + (\delta C)/C]$.

We measured five circular objects, graphed the results, and obtained π = 3.132 ± 0.015 using error analysis techniques. The only physical condition necessary is that the objects be perfectly circular. This experiment allows students to test if three-dimensional space is Euclidean within the limits of the measurements.

In conclusion, we learned a lot trying to discover simple numbers in physical systems. It was $\sqrt{2}$ that led us to thinking how the properties of systems scaled with size. The natural logarithm e made us find a universal ingredient in decaying systems: linearity. The most interesting aspect of these experiments occurs when you don't get the number expected. Then you are challenged to understand your experimental errors and the essential physics that makes these numbers so unique. We hope that your class will find these experiments interesting, and hope they will discover ways to measure these and other special numbers in nature.

Reference

1. See for example Fishbane, Gasiorowicz, and Thornton, *Physics for Scientists and Engineers* (Prentice-Hall, Englewood Cliffs, NJ, 1993), pp. 22–25.