Notes I on Introductory Mechanics
Statics and Describing Motion

These notes are meant to help students understand the basic "physics" behind introductory "Newtonian" mechanics. The ideas are presented in the order in which they are taught in my first year class, and are designed to supplement the text.

The approach we take is somewhat historical. In this section we first study the conditions for static equilibrium for a rigid object: how forces combine such that a rigid object does not move or rotate. The branch of mathematics that is relevant for these topics is Vectors. When we have finished statics, we will learn the language for describing motion.

Mechanics before 1500

In early times the balancing of forces was important for building, and was the first "physics" to be investigated. Later on in the quarter we will analyze mechanical systems precisely. We will discuss the concepts of inertial mass, force and motion, ideas that were discovered in the 1600's. For now, we will consider a simple set of experiments with forces.

Simply speaking, a force is a push (or pull). Let's consider forces due to an objects weight, i.e. the gravitational force due to the attraction of the earth. For this discussion, we limit ourselves to objects that are made of pure elements, i.e. copper, iron, etc. We will make the assumption that for a pure homogenious substance the weight is proportional to the volume of the object. That is, if object A is twice as large object B, and both are made of the same substance, then object A has twice the weight as object B. If one object has a weight of 10 units, then an object of twice (or x times) the volume will have a weight of 20 units (or 10x units). The weights in our weight set are so labeled.

Combining forces in the same direction

In our first experiments we will combine forces on a ring by using pulleys and strings. A string attached to the ring will pass over a pulley. The other end of the string will be attached to a weight. Suppose two weights, each of magnitude 10 units, pulls on the ring to the right (the + direction). How much weight and where should it be placed to balance the two weights of 10 units. Your guess is probably one weight of magnitude 20 units pulling to the left. We will check this result in class. (Your guess was correct). This means that two forces pulling in the same direction add like real numbers. Similarly, we will show that two forces pulling in the opposite direction

subtract. Thus for the case of one dimension, we can assign a (+ or -) to a force to signify the direction (right or left).

Combining forces in different directions
    A forces can act in any direction. What role does the direction of a force play when combining forces? Consider the following experiment:

A force of 40 units pulls on the ring to the east and a force of 30 units pulls on the ring to the north. What force on the ring will balance these two forces?

After doing the experiment in class, we will find that the balancing force is a force that pulls on the ring with an amount of 50 units at a direction of 36.9° S of W. Thus the force of 40 units to the east plus the force of 30 units to the north is equivalent to a single force of magnitude 50 units at an angle of 36.869° N of E. In this case 30 plus 40 equals 50. You probably remember from trig that lengths of 30, 40, and 50 form a right triangle (3-4-5 triangle). Thus, if we represent each force by an arrow, whose length is proportional to its magnitude and whose direction is in the direction of the force, the two forces combine by placing the tail of one at the tip of the other. The resultant force is represented by the arrow along the hypotenuse.
    In lab, you will carry out a number of experiments that demonstrate that any two forces combine using the "arrow" tail-to-tip method described above. The mathematics that best describes how forces combine is the mathematics of *vector addition.* Since a number of physical quantities have the properties of mathematical vectors, we will spend some time in lecture discussing vectors.
    Vectors are discussed in many texts and usually are defined (initially) as something with direction and magnitude. Quantities with direction and magnitude are not necessarily vectors. They must also have certain mathematical properties. An addition operation must be defined, and the sum of two vectors must also be a vector. If $\vec{A}$ and $\vec{B}$ represent any two vectors, then it must be true that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Vectors are also defined over a field, which in our class will be the real numbers. To be an "inner-product" vector space, a scalar product between two vectors must be defined with certain properties. We will talk about this later when we discuss energy.
    The experimental result pertaining to adding "static" forces is summarized by the following experiment:

Experiment: If two forces, $\vec{F_1}$ and $\vec{F_2}$, act on an object that doesn't move, the

resulting force is the same as if the object were subject to the force $\vec{F}$ where $\vec{F}$ is the **vector sum** of the two forces: $\vec{F} = \vec{F_1} + \vec{F_2}$. The same vector addition applies if more than two forces are acting on the object.

This is a wonderful experimental result! Forces didn't have to "add" in such a simple way, but they do. It demonstrates that certain aspects of nature can be understood using geometry (trig). These ideas were developed over 3000 years ago, and helped in the building of magnificant structures. We will do many examples in class demonstrating this remarkable property of forces, but first we review the mathematical properties of vector addition.

Multiplication of a vector by a scalar

Multiplication of a vector by a scalar c changes the magnitude of the vector by the factor $|c|$. If $c > 0$, then the direction remains the same, but if $c < 0$ the direction is reversed.

Unit Vectors

Unit vectors are vectors that have a magnitude of one. In three dimensional cartesian space, one usually defines the unit vector $\hat{i}$ to point in the +x direction, the unit vector $\hat{j}$ to point in the +y direction, and the unit vector $\hat{k}$ to point in the +z direction. Any vector in cartesian 3D space can be expressed as linear combinations of these unit vectors:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{1}$$

where A_x, A_y, and A_z are the coordinates of the tip of the vector. These unit vectors form an orthonormal basis, whose linear combinations span the vector space. The unit vectors are like "building blocks": any vector can be "built" from these three elements.

Vector Addition

We explain vector addition with an example. Let vector $\vec{A}$ have a magnitude of 150 units at an angle of 30 degrees N of E, vector $\vec{B}$ have a magnitude of 100 units at an angle of 45 degrees N of W, and vector $\vec{C}$ have a magnitude of 200 units at an angle of 80 degrees S of E. Find the vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, which is the sum of the three vectors. One could use the tail-to-tip method to add the three vectors. We

3

will show in class that the tail-to-tip method is equivalent to adding the cartesian components after expressing the vectors in terms of the unit vectors. So, let's add the three vectors using the unit vector basis. First, express $\vec{A}$ as a linear combination of the unit vectors $\hat{i}$ and $\hat{j}$:

$$
\begin{aligned}
\vec{A} &= 150cos(30)\hat{i} + 150sin(30)\hat{j} \\
&= 130\hat{i} + 75\hat{j}
\end{aligned}
$$

from the definition of the unit vectors and scalar multiplication. Then, express $\vec{B}$ in terms of the unit vectors:

$$
\begin{aligned}
\vec{B} &= -100cos(45)\hat{i} + 100sin(45)\hat{j} \\
&= -70.7\hat{i} + 70.7\hat{j}
\end{aligned}
$$

Finally, express $\vec{C}$ in terms of the unit vectors:

$$
\begin{aligned}
\vec{C} &= 200cos(80)\hat{i} - 200sin(80)\hat{j} \\
&= 34.7\hat{i} - 197\hat{j}
\end{aligned}
$$

To add the three vectors, one simple adds the components of the unit vectors:

$$
\begin{aligned}
\vec{A} + \vec{B} + \vec{C} &= (130 - 70.7 + 34.7)\hat{i} + (75 + 70.7 - 197)\hat{j} \\
\vec{D} &= 94\hat{i} - 51.3\hat{j}
\end{aligned}
$$

The sum vector $\vec{D}$ can be expressed in terms of the unit vectors, or as a magnitude and direction: $|\vec{D}| = \sqrt{94^2 + 51.3^2} = 107$ units. The vector $\vec{D}$ points in the fourth quadrant, with the angle $tan^{-1}\theta = 51.3/94$, or $\theta = 28.6°$. Since $\vec{D}$ is in the fourth quadrant, $\theta = 28.6°$ S of E. The calculation is summarized in the figure.

The connection that the mathematics of this example has to physics is the following. If an object is subject to three forces: a force of magnitude 150 units at an angle of 30 degrees N of E, a force of magnitude 100 units at an angle of 45 degrees N of W, and a force of magnitude 200 units at an angle of 80 degrees S of E, then the net force on the object is 107 **units at an angle of** $28.6°$ **S of E.** If one wants to balance these three forces, then one needs a force of 107 units in the opposite direction at 28.6° N of W.
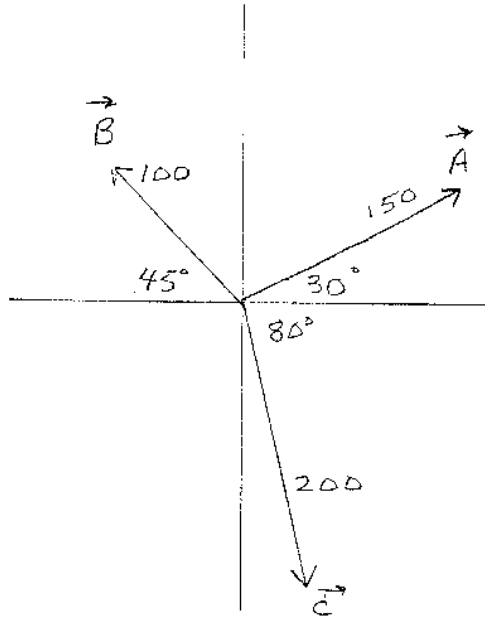
$150 \sin 30° = 75$

$150 \cos 30° \approx 130$

$100 \sin 45° \approx 70.7$

$100 \cos 45° \approx 70.7$
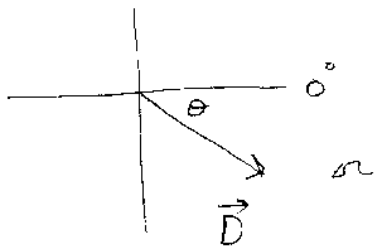
$200 \sin 80° \approx 197$

$200 \cos 80° \approx 34.7$

	x-comp	y-comp
$\vec{A}$:	+130	+75
$\vec{B}$:	−70.7	+70.7
$\vec{C}$:	+34.7	−197
$\vec{A}+\vec{B}+\vec{C}$:	94	−51.3

$\vec{A} = 130\,\hat{\imath} + 75\,\hat{\jmath}$

$\vec{B} = -70.7\,\hat{\imath} + 70.7\,\hat{\jmath}$

$\vec{C} = 34.7\,\hat{\imath} - 197\,\hat{\jmath}$

$\vec{D} = 94\,\hat{\imath} - 51.3\,\hat{\jmath}$

$|\vec{D}| = \sqrt{94^2 + 51.3^2} \approx 107$

$\theta = \tan^{-1}\frac{51.3}{94} \approx 28.6°$

or $\quad 107$ at $(360 - 28.6)°$

107 at $331.4°$

Expressing the vectors using unit (or basis) vectors is probably the most convenient way to perform operations with them. In general, if $\vec{A} = A_x\hat{i} + A_y\hat{j}$, and $\vec{B} = B_x\hat{i} + B_y\hat{j}$, then the sum $\vec{A} + \vec{B}$ equals

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \tag{2}$$

for subtraction, replace the "+" with a "-". Using unit (or basis) vectors is particulary useful when adding (or subtraction) more than two vectors.

Some final points on vectors and vector addition:

1. It is important to remember that **for a complete description of a vector in two dimensions, two numbers are needed**: a magnitude plus direction, two components, etc. For example, the vector $\vec{D}$ above can be expressed as: $94\hat{i} - 51.3\hat{j}$, or 107 units at a direction 28.6° S of E, or 107 units at an angle of 331.4° clockwise from the x-axis.

2. Here we have limited our applications to two and three dimensional vectors. These ideas can be generalized to any number of dimensions (including infinity). For vector spaces with dimension greater than 3, it is not always useful to think of the direction of a vector. The generalization of the ideas presented here is the branch of mathematics called Abstract Linear Algebra.

3. In order for a quantity to be a vector, the quantity must add (or combine) according to the rules of vector addition. Not all quantities with magnitude and direction add accoring to the rules of vector addition. An example of such a quantity is rotations. Any rotation in three dimensions can be represented by an arrow with magnitude and direction: the direction is the axis of rotation, and the magnitude is the amount of rotation (e.g. in radians). Let A be a rotation about the x-axis of $\pi/2$ radians, and B be a rotation about the y-axis of $\pi/2$ radians. You can show by rotating your physics book that $\vec{A} + \vec{B} \neq \vec{B} + \vec{A}$. That forces add according to the laws of simple vector addition is remarkable.

There are a number of quantities in classical physics that behave as vectors: displacement, velocity, acceleration, force, the electric field and the magnetic field to name a few. You will encounter these during your first year of physics. Although they represent different physical quantites, they all add like the method described here. Vector addition pertains to many different physical quantites, it is a "universal mathematics", and its importance cannot be understated.

Torque: twisting force

We all have experience tightening a bolt. If we want it tight, we use a wrench. To apply a large amount of "twisting force", we hold the wrench as far as we can from the bolt and push with a force perpendicular to the wrench arm. The name we call "twisting force" is **torque**. In order to determine the amount of torque, we need to specify three things:

1) The axis of rotation. In the case of the bolt, this is where the bolt is.

2) The force that is doing the twisting. (The force, direction and magnitude, that you apply to the wrench).

3) The location of the force. (How far from the bolt that you apply the force.)

When we speak of torque, we usually say "the torque *about* the axis (or point) O *due* to the force F *applied* at the location P."

All three things matter in determining the torque. We know from experience that a larger force produces a larger torque, and the further away from the axis that the force is applied, the larger the torque, but what is the correct mathematical formula for torque? We will show in class that:

torque $= |\vec{r}|$ times the component of the force perpendicular to $\vec{r}$

where $\vec{r}$ is the vector from the axis to the location where the force is applied. The component of the force perpendicular to $\vec{r}$ is the component of the force that does the twisting. The component of the force, $\vec{F}$, that is parallel to $\vec{r}$ just pushes on the bolt. This component doesn't cause any twisting about the axis. The symbol for torque used in most physics texts is τ:

$$|\vec{\tau}| = |\vec{r}|F_\perp \tag{3}$$

If we let θ be the angle between the vector $\vec{r}$ and the vector $\vec{F}$, the torque equation can be written as:

$$|\vec{\tau}| = |\vec{r}||\vec{F}|sin(\theta) \tag{4}$$

Remember, the vector $\vec{r}$ goes from the axis of rotation to the location at which the force is applied.

Is torque a vector, a scalar, or something else? Torque certainly has a magnitude, which is the amount of twisting force. We can also associate a direction with torque:

The axis of rotation of the twist. The axis of rotation is in the direction perpendicular to both the force $\vec{F}$ and the position vector $\vec{r}$. Thus, we can assign a vector to a particular torque: The magnitude of the vector is the amount of twisting force, $|\vec{r}||\vec{F}|sin(\theta)$; and the direction is the direction perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. There is one more decision to make: which direction along the axis of the torque corresponds to a clockwise twist, and which direction to a counter-clockwise twist? Since $sin(-\theta) = -sin(\theta)$, the vector for clockwise twist will be in the opposite direction to the one for counter-clockwise twist. The convention that is agreed upon around the world is the following: the direction of the torque vector is the direction that the bolt would move (for normal "right-handed" bolts). Using your right hand, if your fingers curl in the direction of the twist, your thumb points in the direction of the torque.

We will show that torques add according to the rules of vector addition, so it is a true vector. In mathematics, there is an operation between vectors that yields another vector. It is the cross product and we can write the torque vector as $\vec{r}$ cross $\vec{F}$:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{5}$$

In this course, we will only solve problems in which all the location $\vec{r}$ vectors and all the force $\vec{F}$ vectors lie in the same plane. Thus, we will not need to carry out the cross product explicitly using vectors. We will only do problems in which the torques produce a clockwise or counter-clockwise twist, and consquently the torque vectors are either in or out of the page. Nonetheless, to prepare you for more advanced courses, we review some of the mathematical properties of the vector cross product.

Vector Cross Product

The cross product of vector $\vec{A}$ and $\vec{B}$ is a vector. The magnitude of $\vec{A} \times \vec{B}$ equals the magnitude of $\vec{A}$, $|\vec{A}|$ times the magnitude of $\vec{B}$, $|\vec{B}|$, times the sin of the angle between them:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin(\theta) \tag{6}$$

where the angle θ is the angle between vector $\vec{A}$ and vector $\vec{B}$. The angle goes from $\vec{A}$ to the vector $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is determined by the "right hand" rule described above. Note that since $sin(-\theta) = -sin(\theta)$,

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \tag{7}$$

Note that the cross product is maximized when $\vec{A}$ is perpendicular to $\vec{B}$, and zero when θ is zero or $180°$. The unit vectors have simple cross product properties:

$$\begin{aligned}
\hat{i} \times \hat{j} &= \hat{k} \\
\hat{j} \times \hat{k} &= \hat{i} \\
\hat{k} \times \hat{i} &= \hat{j} \\
\hat{i} \times \hat{i} &= 0 \\
\hat{j} \times \hat{j} &= 0 \\
\hat{k} \times \hat{k} &= 0
\end{aligned}$$

for a right handed coordinate system.

We can express $\vec{A}$ and $\vec{B}$ in terms of the unit vectors: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$. If we use the above forumlas for the cross products of the unit vectors, after "foil"ing we have:

$$\begin{aligned}
\vec{A} \times \vec{B} &= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} \\
&\quad + (A_x B_y - A_y B_x)\hat{k}
\end{aligned}$$

This formula can be succinctly written in terms of the determinant of a matrix:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{8}$$

Condition for Static equilibrium for a rigid object

In summary, there are two conditions for static equilibrium. In order for an object to stay at rest, the vector sum of the forces on the object must be zero. This is all one needs if the object is a point particle. However, if the object has size, the location of the forces on the object is important. Eventhough the sum of the forces is zero, the object might rotate. In order for the object to stay stationary and not rotate, the sum of the torques must be zero about any axis. These are the two conditions for an object to remain "static", i.e. not translate and not rotate:

$$\begin{aligned}
\sum_i \vec{F}_i &= 0 \\
\sum_i \vec{\tau}_i &= 0
\end{aligned}$$

9

where the torque sum can be taken about any axis. Note that for the first requirement, $\sum_i \vec{F_i} = 0$, the location of the forces do not enter in the calculation. However, for the second requirement, the location of the forces are very important. Just like in Real Estate, the three most important things are Location, Location, and Location. With torque, it is just Location, Location: the Location of the axis and the Location of the forces.

In class we will do many examples of objects in static equilibrium. In evaluating the torque sum, a proper choice of axis can reduce the mathematical complexity.

One Dimensional (straight-line) Frictionless Environment

Objects are not always stationary. They move. In order to formulate the laws of motion, we need to come up with a useful language. It turns out that the laws of motion are most easily described in terms of instantaneous rates of change. The mathematics of calculus was key to understanding the laws motion, and we start with the language for describing motion.

Describing Motion: Position, Time, Velocity

To describe motion of a point particle in one "straight-line" dimension one needs to come up with a measure of two quantities: position (or distance) and time.

To measure **position**, we set up a coordinate system, which in one dimension is just a straight line with a reference point chosen as the origin. Once a direction for positive distance and a unit length have been decided upon, equal lengths can be marked on the line. A particles position is specified by a real number (+ or -) indicating its location on the line. Two systems of units are used in the class, MKS and British. Length is measured in meters in the MKS, and in units of feet in the British system.

To measure **time**, we need an instrument (a clock) that can produce equal time intervals. Whereas we have a physical feeling for equal distances from our hands, arms and eyes, equal time intervals are difficult to get a feeling for. Physiologically a good movie lasting as long as a physics lecture might seem much shorter in time. Therefore we can't rely on our senses to judge equal time intervals. One could use a pendulum or other oscillating device. However, using the motion of a physical system to determine equal time intervals and then using this clock to understand motion itself might bias our description and laws of physics. We might wonder if our method of measuring time is making our equations of physics to complicated. It would be nice to have a clock that does not rely on a physical system. We will discuss such a possibility with Newton's first law of motion. In order not to get caught up in circular

arguments and deep philosophy, in this class we will accept that clocks can be made that give equal time intervals. As discussed in the text, the accepted definition of a second is 9192631770 oscillations of a particular transition in the Cesium atom. Although the atom is a physical system, we obtain a workable definition of time for developing physics theories. The interested student should take our modern physics course, where Einstein's theory of special relativity addresses these problems.

Once we have established distance and time units, we can define a **position function**, $x(t)$. The position function is just the position x for the particle as a function of time, t.

A particles **displacement** between the times t_1 and t_2 is just equal to $x(t_2) - x(t_1)$, where $t_2 > t_1$. If the displacement is positive, the particle has moved to the right (+ direction) from the time t_1 to the time t_2. If the displacement is negative, the particle has moved to the left (- direction). To talk about displacement, you need to refer to *two times*.

Velocity is a measure of how fast an object is moving. **Average velocity**, $\bar{v}_{t_1 \to t_2}$, is defined to be the displacement/(time interval):

$$\bar{v}_{t_1 \to t_2} \equiv \frac{x(t_2) - x(t_1)}{t_2 - t_1} \tag{9}$$

As with displacement, to calculate the average velocity of an object one needs to specify the *two* times, t_1 AND t_2. Average time is not a particularly good quantity to express the laws of mechanics in a simple way. For understanding the laws of physics in a simple way, the velocity at an instant, instantaneous velocity, is a much better quantity. **Instantaneous velocity** at the time t_1, $v(t_1)$, is defined to be:

$$v(t_1) \equiv \lim_{t_2 \to t_1} \frac{x(t_2) - x(t_1)}{t_2 - t_1} \tag{10}$$

This can also be written as:

$$v(t_1) \equiv \lim_{\Delta t \to 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} \tag{11}$$

which you will recognize as the first derivative of $x(t)$ evaluated at the time t_1: $v(t_1) \equiv (dx/dt)|_{t=t_1}$. A nice thing about instaneous velocity is that it is defined at a single time, t_1. For this reason it is a better quantity to use than average velocity to try and describe how nature behaves. If the instantaneous velocity is positive (negative) then the particle is moving to the right (left) at that instant.

A particle's **speed** is defined to be the magnitude (or absolute value) of the instantaneous velocity.

Special Case of Constant Velocity

In some special cases, a particles velocity may be constant for a long period of time. If this is so, then the position function takes on a simple form. Let v_0 be the velocity of the particle at the time when we start our clocks, $t = 0$. We will often use the subscript 0 to label the value of a variable at time $t = 0$. For a particle moving with a constant velocity, v_0 will be the velocity for all times t. From the definition of velocity: $(x(t) - x(0))/(t - 0) = v_0$. If we define $x(0) \equiv x_0$, we have the simple relationship for $x(t)$:

$$x(t) = x_0 + v_0 t \qquad (12)$$

Often one writes $x(t)$ as just x, so the equation becomes $x = x_0 + v_0 t$.

Under what situations will an object move with a constant velocity in a straight line? A person walking in a straight line down the street with a constant speed, or a car driving down a straight length of road with a constant speed are some examples that come to mind. However, there is a very important case related to a fundamental Law of Motion.

Consider a "reference frame" which is a box floating in space far from any other objects. If you were to go inside this box, floating in space, you would feel "at rest". You could not sense that you were moving. You set up a coordinate system to measure position (one dimensional) and time. Suppose a particle had an initial velocity v_0 with respect to your coordinate system. There are no forces on the particle. What would happen to the particle in time? Would it come to rest, or continue to move with the velocity v_0 in a straight line? Newton and Galileo realized that the particle would continue to move with the velocity v_0. If the particle was at rest ($v_0 = 0$) it would remain at rest. This property of motion is refered to as Newton's first Law of Motion:

If there are <u>no forces</u> acting on an object, an object at rest remains at rest and an object in motion continues in a state of uniform motion

This idea might seem simple to us, but at the time it was proposed it was profound. It was believed that the natural state of an object was at rest, and that objects that were moving came to rest on their own. Newton's first law of motion applies to reference frames that are floating in space. Only in these frames will an object that

is released at rest in "mid air" stay at rest. The name we give to reference frames for which this law (Newton's first law) of motion holds is an **inertial reference frame**.

Imagine two reference frames floating in space. Suppose someone named Bill was in one, and George in the other. Suppose that George observed that Bill was moving in the + direction with a *constant velocity* $+v$. George would feel at rest and say that Bill was moving to the right with a constant velocity. Bill, however, would also feel at rest and say that George is moving to the left with a constant velocity $-v$. Who is correct? Both are. Each of these frames is an inertial reference frame. Bill feels at rest, and so does George. There is something very special about reference frames floating in space with a constant velocity with respect to each other. They are all inertial reference frames and have the following properties:

1. A reference frame moving with a constant velocity with respect to an inertial frame is also an inertial reference frame.

2. In an inertial reference frame, one "feels" at rest.

3. There is no experiment that one can do in an inertial reference frame to determine the velocity of the reference frame.

4. The laws of physics take on the same form in all inertial reference frames.

5. There is no absolute reference frame.

The equivalance of inertial reference frames is a fundamental property of physics, and is the basis of Einstein's theory of special relativity. It is a wonderful property of nature, and one can marvel at its simplicity.

A final note on Newton's first law is that it allows one to define equal time intervals independent of a physical system. Here is how to do it: Set up your "x" axis, pick an origin, and a unit length. Use your unit length to mark on your "x" axis equal distances. Then set an object (with no forces) in motion. It will float along your "x" axis. A time interval occurs each time that it passes a mark. According to Newton's first law the time intervals will be equal.

Describing Motion: Acceleration

Objects don't always move with constant velocity, velocities change. The change in velocity per unit time is called acceleration. The average acceleration, $\bar{a}_{t_1 \to t_2}$,

between time t_1 and t_2 is defined to be

$$\bar{a}_{t_1 \to t_2} \equiv \frac{v(t_2) - v(t_1)}{t_2 - t_1} \tag{13}$$

As with average displacement and average velocity, one needs *two times* to calculate the average acceleration. Since two times are needed, the average acceleration is not a good quantity to describe the laws of physics. A more useful quantity is the instantaneous acceleration, which is defined in the same way as instananeous velocity. One takes the limit of the average acceleration as t_2 approaches t_1:

$$a(t_1) \equiv \lim_{t_2 \to t_1} \frac{v(t_2) - v(t_1)}{t_2 - t_1} \tag{14}$$

Instantaneous acceleration is also written as:

$$a(t_1) \equiv \lim_{\delta t \to 0} \frac{v(t_1 + \delta t) - v(t_1)}{\delta t} \tag{15}$$

The limit on the right side is just the first derivative of the velocity evaluated at the time t_1. So $a(t_1) \equiv dv/dt|_{t=t_1}$. Often the t_1 is replaced by t, and one writes $a \equiv dv/dt$. A nice property about instananeous acceleration is that it is determined at *one time*. The acceleration is just the second derivative of position with respect to time: $a = d^2x/dt^2$.

One could also consider more derivatives such as da/dt, d^2a/dt^2, etc... to describe motion. We need to rely on experiments to determine the simplest way to relate interactions (forces) and motion.

If the position function, $x(t)$, is known, it is easy to find $v(t)$ and $a(t)$ by differentiation. If one knows $a(t)$, one needs to integrate with respect to t to find $v(t)$. To find $x(t)$, one needs to integrate $v(t)$ with respect to t. A simple case that is discussed in many texts is the special case of motion with constant acceleration. Suppose the acceleration of a particle is constant, label it a_0. Then, $dv/dt = a_0$. Multiplying both sides by dt and integrating we have $\int_0^t dv = \int_0^t a_0 dt$, which gives $v(t) - v_0 = a_0 t$ where v_0 is the velocity at $t = 0$, $v(0)$. This is often written as $v = v_0 + a_0 t$, where v means $v(t)$.

To solve for $x(t)$ for the case of constant acceleration requires one more integration. From $dx/dt = v(t) = v_0 + a_0 t$, one can multiply both sides by dt and integrate: $\int_0^t dx = \int_0^t v_0 dt + \int_0^t a_0 t dt$. After integrating, one obtains: $x(t) - x(0) = v_0 t + a_0 t^2/2$. This equation is often written as $x = x_0 + v_0 t + a_0 t^2/2$, where x means $x(t)$.

Summarizing for the *special case of constant acceleration*:

$$v = v_0 + a_0 t \tag{16}$$

$$x = x_0 + v_0 t + \frac{a_0}{2} t^2 \tag{17}$$

Eliminating t from these two equations gives:

$$v^2 = v_0^2 + 2a_0(x - x_0) \tag{18}$$

In general, motion that has constant (non-zero) acceleration for extended periods of time is fairly rare. Under certain circumstances however, the motion of an object can be *approximately* one of constant acceleration for a period of time. These special cases include situations where the net force is *approximately* constant (e.g. free fall near the earth's surface in the absence of air friction, rolling or sliding down a ramp with no friction, ...). Since the equations for position and velocity are simple for motion of constant acceleration, many texts use examples and problems of this special case to test the student's understanding of these concepts.

One final note: the sign's of $x(t)$, $v(t)$, and $a(t)$ are not related to each other. The acceleration can be in the negative (positive) direction eventhough the velocity is positive (negative), etc. The sign of a is in the direction of the change of v.

Describing Motion in 2 and 3 Dimensions

The extension of velocity and acceleration to 2 and 3 dimensions is straighforward, since displacements add like vectors. We can define a position vector, $\vec{r}(t)$, a velocity vector, $\vec{v}(t)$, and an acceleration vector, $\vec{a}(t)$ as follows

$$
\begin{aligned}
\vec{r}(t) &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \\
\vec{v}(t) &= \frac{d\vec{r}(t)}{dt} \\
&= v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k} \\
\vec{a}(t) &= \frac{d\vec{v}(t)}{dt} \\
&= a_x(t)\hat{i} + a_y(t)\hat{j} + a_z(t)\hat{k}
\end{aligned}
$$

If we pick our coordinate system properly, sometimes we can change a three dimensional problem into three one dimensional problems. An important point to mention is that in general **the three vectors $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ do not necessarily point in the same direction**.